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On a result of Bernstein

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According to Bernstein [1, p. 90] the smallest uniform error obtained in approximating $(1-x)^{-1}$ on $[-\frac{1}{2}, \frac{1}{2}]$ by polynomials $\sum_{k=0}^{n} c_k x^k$, $n \geqslant 0$, c_k integers, $c_n = 1$, is 2^{-n}. A related result is the following

THEOREM. *Let $a > 1$, m, n integers $\geqslant 0$, and either m is odd or $m \leqslant n$. Then*

$$\min_{Q \in \Pi_m} \left\| \frac{1}{1-x^{m+1}} - \frac{\sum_{i=0}^{n} x^i}{Q(x)} \right\|_{L^\infty[-1/a,\, 1/a]} = a^{m-n}(a^{m+1}-1)^{-1}$$

where Π_m denotes the class of all polynomials Q of degree m whose coefficients are positive integers, with $Q(x) > 0$ throughout $[-1/a, 1/a]$.

Proof. For $0 \leqslant x \leqslant a^{-1}$,

$$0 \leqslant \frac{1}{1-x^{m+1}} - \frac{\sum_{i=0}^{n} x^i}{\sum_{i=0}^{m} x^i} = \frac{x^{n+1}}{1-x^{m+1}} \leqslant \frac{a^{m-n}}{a^{m+1}-1}$$

as $x^{n+1}(1-x^{m+1})^{-1}$ is increasing in $[0, 1)$. For $-a^{-1} \leqslant x < 0$, n odd,

$$0 < \frac{1}{1-x^{m+1}} - \frac{\sum_{i=0}^{n} x^i}{\sum_{i=0}^{m} x^i} \leqslant \frac{a^{m-n}}{a^{m+1}-1}$$

as $x^{n+1}(1-x^{m+1})^{-1}$ is decreasing in $(-1, 0)$. Similarly for $-a^{-1} \leqslant x < 0$, n even,

$$0 < \frac{\sum_{i=0}^{n} x^i}{\sum_{i=0}^{m} x^i} - \frac{1}{1-x^{m+1}} \leqslant \frac{a^{m-n}}{a^{m+1}-1}.$$

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Hence,

$$\left\| \frac{1}{1-x^{m+1}} - \frac{\sum_{i=0}^{n} x^i}{\sum_{i=0}^{m} x^i} \right\|_{L^\infty[-1/a,\, 1/a]} = \frac{a^{m-n}}{a^{m+1}-1}.$$

On the other hand, let $Q \in \Pi_m$. Then

$$\left\| \frac{1}{1-x^{m+1}} - \frac{\sum_{i=0}^{n} x^i}{Q(x)} \right\|_{L^\infty[-1/a,\, 1/a]} \geqslant \frac{1}{1-(1/a)^{m+1}} - \frac{\sum_{i=0}^{n} (1/a)^i}{\sum_{i=0}^{m} (1/a)^i} = \frac{a^{m-n}}{a^{m+1}-1}.$$

Reference

1. A. F. Timan, "Theory of Approximation of Functions of a Real Variable," The MacMillan Co., New York, 1963.

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